

Mathematical Textiles

The Use of Knot Theory to Inform the Design of Knotted Textiles

Nithikul Nimkulrat*, Janette Matthews**

* Loughborough University, UK, N.Nimkulrat@lboro.ac.uk

** Loughborough University, UK, J.Matthews@lboro.ac.uk

Abstract: This paper reports on an ongoing practice-led research project examining the relationship between mathematical knot theory and knotted textiles, i.e., how mathematics may be used to characterize knotted textiles and how mathematics learners and textile designers can mutually benefit from this relationship. The research questions include: (1) whether craft and mathematical knots share comparable characteristics; (2) whether knot theory can examine the mathematical properties of knotted textile structures; and (3) how knot theory can facilitate the conceptualization, design and production of three-dimensional textiles. This paper focuses on the first phase of the research process, which commences with the mathematical characterization process which enables three-dimensional knotted textiles previously created by the author to be considered, e.g., what a knot is, how it is formed, what makes two knots equivalent, what composite knots (two or more knots together) are, what effect the spaces between and within knots have, what influence material characteristics (elastic, flexible, or rigid) have on knots, etc.

Key words: *Craft, Experiential Knowledge, Knot, Knot Theory, Mathematics, Textiles*

1. Introduction

[A] knot is not referential but synthetic, in relating inextricably the texture of its surface to the logic of binding. Unlike the open mesh of the looped string, the knot does not hint at what lies beneath its surface, but is itself to be discovered beneath its own surface. The knot is all that is to be seen. The knot is the knowledge, a knowledge of the linking of things, material and mental, that may as well exist apart. [12]

Knots have been commonly used for nearly two thousand years in many cultures as decorative elements [11]. Knotting or macramé is a craft technique commonly used in textile design. While free ends are essential for knotting textile work, “no string with free ends can be knotted” in mathematical knot theory, or topology [4]. This difference between craft and mathematical knots generates a question as to whether both types of knots share any similarities. It is hoped that the answer to this question may establish a transdisciplinary link between mathematics and textile design.

This paper reports on an ongoing practice-led research project that is a collaboration between a textile practitioner-researcher (Nimkulrat) and a textile practitioner with a first degree in mathematics (Matthews). The project aims to provide an insight into the dynamic relationship between mathematical knot theory and knotted textiles, examining how mathematics may be used to characterize knots and how mathematics learners and textile designers can mutually benefit from this relationship. The research questions include: (1) whether craft and mathematical knots share comparable characteristics; (2) whether mathematical knot theory can be used to

examine the mathematical properties of knotted textile structures; and (3) how knot theory can facilitate the conceptualization, design, and production of three-dimensional textiles.

Based on the aforementioned viewpoint of “a knot is a knowledge”, the objective of this project is to illuminate (1) the way in which tacit craft knotting skills embodied in the process of research can be articulated and (2) whether mathematical knot theory can be used as a means and method to transfer and replicate this tacit knowledge. The study explores the role of the researcher’s professional knowledge and the different ways in which it can be utilized and communicated within the framework of research – an investigation into modes of communication and exchange for experiential and procedural knowledge.

2. An Overview of the Relationship between Art, Design, and Mathematics

Mathematics reveals facts inherent in nature, e.g., rotational symmetry of flowers and, fractals the system of arteries in the human body. Applications of mathematics can be found not only in biology, chemistry, and physics, but also in art, music, design, and architecture. Mathematical concepts have been adopted in the creation of various forms of art, ranging from geometrical sculptures to computer graphics. In return, art can visually convey the phenomena of complex mathematical concepts to a wide audience, enabling people to understand things surrounding or even inside them. The connections between art and mathematics have increasingly become evident since the emergence of the Bridges Conference series and its mathematical art exhibitions in 1998 and the *Journal of Mathematics and the Arts* in 2007 [5,6,17].

In the field of textile art and design, mathematics seems to have advanced the design of textiles, especially in technical fields, e.g., models for entangled fibers [13]. Mathematics can also benefit the textile industry through the use of exact calculation [21,22] and in production. On the other hand, textiles can facilitate the understanding of mathematics, thus influencing curriculum development, teaching materials production and fundamental mathematics education [7,8]. Harris was a mathematician who developed ways of teaching mathematics through looking at domestic textile craft objects. She used textiles to visualize mathematical concepts, such as symmetry, pairs, patterns, sets, lattices, tension, nets and solids, visual, tactile, and three-dimensional. She initiated “Common Threads”, a touring exhibition in England between 1991 and 1992, to demonstrate the wealth of tacit mathematical thinking in the traditional work of women around the world and to question male dominance in both mathematics and work. A more contemporary example is the Japanese fashion designer Issey Miyake, who employed the mathematics of folding for the development of the 132 5. ISSEY MIYAKE series [10].

Knot theory is one of the mathematical concepts whose applications in the arts can be seen throughout the history of art in many cultures, such as Babylonian, Egyptian, Greek, Byzantine, Celtic, and Chinese art [11]. Although knotting and representations of knots have widely been utilized in various art forms, little evidence has been found of the application or study of mathematical knot theory. This is in contrast to other mathematical concepts e.g., geometry, Fibonacci numbers or the Golden Ratio. No evidence has been found of investigations into knot theory and links to aesthetics in textile structures and to the materiality of textile artifacts. The question of how this concept can facilitate the conceptualization and production of textiles remains open.

3. Methods: Collaborative Approach of Textile Practice and Mathematical Knot Theory

In order to answer the research questions aforementioned in the introduction, the process of inquiry is carried out in three areas: (1) the mathematical characterization of craft knots and the development of mathematical models to stimulate design ideas that may not have occurred otherwise; (2) the development of communication methods (e.g., taxonomy, symbols, etc.) for the new knowledge about the relationship of knot theory and knotted textiles to engage and be accessible to the two communities: art, craft & design, and mathematics; and (3) the development of a design research methodology that incorporates mathematical characterization, analysis, and modeling.

Published research on mathematics and the arts has been reviewed with a focus on the three research areas. Firstly, the mathematical characterization of craft knots and design ideas and its use to analyze the arts in many cultures was examined. Two studies were found. A mathematical approach was utilized to demonstrate that a circular knotwork pattern found in historical stones in Scotland can be mathematically categorized, by describing the knot as having an inner and outer region. Using only three parameters to characterize the knotwork, the number of strands may be calculated [2]. Another example is a study examining the Jordan Curve Theorem which shows through hand-drawings that a Jordan Curve or the unknot (a continuous curve with no crossings) when distorted may be used to create a narrative for design [16].

The second research area regarding communication methods for knowledge about knotted textiles characterized by knot theory looks into how previous research has developed methods for communicating mathematical characterization. For example, a drawing of Celtic knots is examined from the point of view of a mathematician and a designer [3]. The study demonstrates by the use of color and mathematical analysis that the design may be aesthetically developed for use with other materials. It shows that mathematical analysis can lead to the understanding of the structure and then to aesthetic development.

Thirdly, the methodology for art, craft and design incorporating mathematics touches upon the applications of mathematics in the arts and vice versa. There are several examples, e.g., the incorporation of mathematical modeling into design production to develop a methodology for rapid manufacturing, such as 3D printing [9], and the textile manifestation of mathematical concepts that may be used to teach mathematics such as Harris's teaching resources [7,8] and the use of crochet to represent hyperbolic planes. [20].

The research project is structured into three phases. The first phase commences with the mathematical characterization process which enables the previous three-dimensional knotted textiles of Nimkulrat to be considered, e.g., what a knot is, how it is formed, what makes two knots equivalent, what the knot characteristics such as the number and direction of strands are, what composite knots (two or more knots together) are, what effect the spaces between and within knots have, what influence material characteristics (elastic, flexible, or rigid) have, etc. Knotting or macramé is the technique Nimkulrat has used in her textile practice for nearly a decade. Her textile knot practice was integrated into her completed PhD research as the main method for inquiry [15]. Through knotting paper string into three-dimensional artifacts, Nimkulrat examines the expressivity of this textile material. This first phase is carried out in collaboration with Matthews who is a textile practitioner with a first degree in mathematics. While Matthews categorizes and analyzes Nimkulrat's textiles, appropriate characterization models, taxonomy and symbolic languages to describe the knots that are employed are developed. Both Nimkulrat and

Matthews then use the newly developed mathematical models to raise questions and stimulate ideas for the design development of new knotted textiles.

The second phase will iterate the cycle performed in the first phase. It will continue from the knotted textile artifacts produced in the first phase which will undergo mathematical analysis and categorization. The characterization will lead to the development of mathematical models that may be used by the artist-researchers to improve the designs and production of knotted textiles. The final phase will examine the overall interaction between the mathematical categorization process and the knot design development process. In this phase, a design research methodology that includes the communication of the research findings to two diverse communities: art, craft & design, and mathematics will be developed.

This paper focuses on the first phase of the research process, which is the current stage of the project. An individual piece from the installation namely *The White Forest* (Figures 1 and 2) created by Nimkulrat was used as a case study for the mathematical characterization process which will be described in the next section.



Figures 1 and 2 *The White Forest* installation and the knotted structure of an individual piece taken for mathematical characterization.

4. Mathematical Characterization Process

This section considers how textiles previously created by Nimkulrat may be considered in terms of mathematical knot theory. Discussion between Nimkulrat and Matthews revealed that three textile knot types are used exclusively in Nimkulrat's textile practice. Structures in Nimkulrat's work result from the same knots being used in combination, e.g., in repeats. As may be seen in Figure 2, material use and knot spacing is consistent. Isolating a group of knots from the work shown in Figure 2, one of these knot types is considered (Figure 3). The other two knot types will be considered and reported in due course.

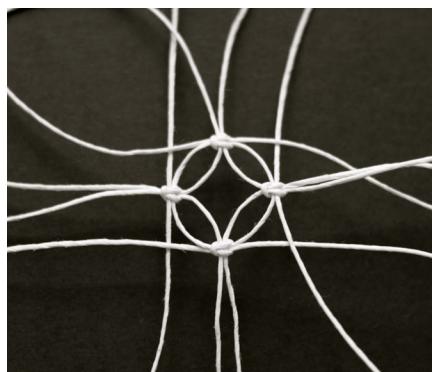


Figure 3 Image of Nimkulrat's flower net isolating one flower and showing four identical knots.

Initial steps were for Nimkulrat to communicate how the knots in Figure 3 were executed. This was achieved by demonstration, discussion, and replication by Matthews. Detailed notes, photographs and sketches were made.

The flower net shown in Figure 2 is formed by tying the same knot at regular intervals. Using eight strands, four knots form a knot flower (Figure 3). Each knot is formed using four strands. In order to document the process, a single knot is considered. The process may be described with the following steps (Figure 4).

1. Start with four strands a, b, c, d (Figure 4, Step 1)
2. Pass strand a over b and c and under d (Figure 4, Step 2)
3. Pass strand d under c and b and up through the loop made by strand a (Figure 4, Step 3)
4. Tighten. Strands positions are now d, b, c, a (Figure 4, Step 4)
5. Pass a over c and b and under d (Figure 4, Step 5)
6. Pass d under b and c and up through the loop formed by a. (Figure 4, Step 6)
7. Tighten. Strands are now in the original positions a, b, c, d (Figure 4, Step 7)

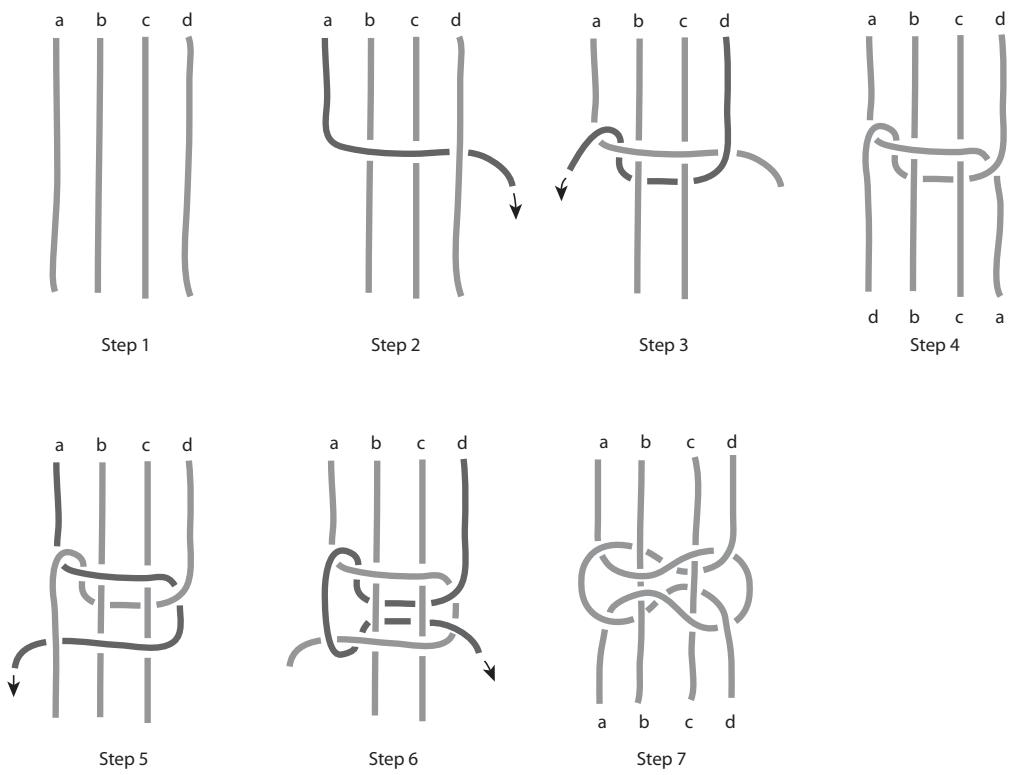


Figure 4 Tying one of the knots in Figure 3 using four strands – a, b, c, d.

The finished knot may be viewed in Figure 5 (left). Readers familiar with the tying of knots and their names will observe that this is a reef knot with two additional vertical stands passing through the center of the knot and trapped in such a way that the knot remains flat. The knot has the same appearance from both the front and the back. If the same method is followed starting with strand d in Step 2 above and a in Step 3, the same knot type will be formed but the raised loop will show on the left of the knot. (Figure 5, right).

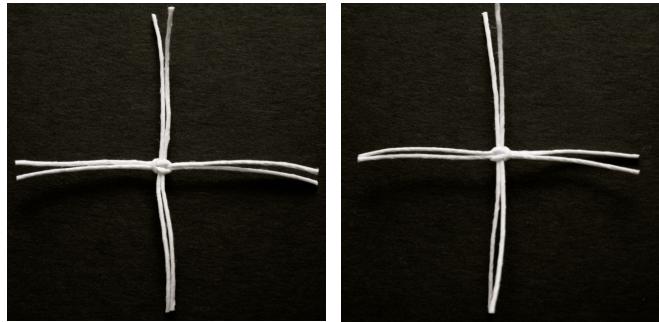


Figure 5 Left, photo of Nimkulrat's knot back and front and right, photo of alternative orientation starting with d instead of a.

It is interesting to note that Nimkulrat does not articulate the making in this way. Knots are known by names such as ‘left hand over’ or ‘right hand starting’ and the knots are tied intuitively employing tacit knowledge [14]. Some time and several attempts were required to unpick and replicate the process.

In order to characterize this knot in terms of mathematical knot theory, it is now necessary to consider some of the properties of mathematical knots.

A mathematical knot, or a representation of a knot, may be defined as a closed curve in space [19]. If the curve defining the knot is imagined as a fine, flexible, and elastic thread, the curve may be moved in a continuous way without cutting and rejoining. All changes of position of the curve will result in the same knot by definition however the representations of the knot may become unrecognizable [19]. Mathematical knot theory is concerned with determining whether two different representations are representations of equivalent (the same) knots.

To explain this further, it is useful to consider a simple knot. Take a piece of string, pass the right end (r) over the left end (l) and under. Leave the knot loose and join the ends. The knot shown in Figure 6 left is achieved with the dot representing the join. This knot may be rearranged into the form on the right known as a trefoil knot.

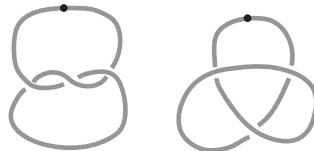


Figure 6 Left, tie a knot – right string over left string and under, join ends, rearranged into trefoil on right.

As this project is concerned with examining the similarities between textile knot practice and mathematical knot theory, it is interesting to consider what has been learned at this stage. These points are summarized in Table 1 below.

- A mathematical knot is defined as a continuous curve and as such does not have loose ends. Knots created through textile practice do have loose ends, which may sometimes be joined. It is also possible to cut a knot and open it up to form a braid (vertical strings tangled in a certain way). A knot can be made

from a braid through joining the upper ends to lower ends [19]. Still under consideration is the precise classification of Nimkulrat's textile practice and the researchers plan to return to the mathematical theory of braids.

- The form of knots created through textile practice is influenced by the characteristics of the materials used e.g., thickness of strand, elasticity, stiffness, or pliability. Mathematical knot theory is not concerned with this. In the definition of a mathematical knot, the cross-section is deemed to be a single point [1].
- The appearance of knots used in textile practice is determined by the tension or tightness of a knot. In mathematical knot theory, knots may be represented by a diagram showing crossings. These are not scale drawings and both tight and loose textile knots will have equivalent mathematical knot representations.
- In mathematical knot theory, two knots are considered equivalent if, after simplification such as the removal of any unnecessary crossings, they have the same number of crossings and orientation. In figure 7, knots a and b are not equivalent. The trefoil knot (a) has three places where the string crosses and the figure eight knot (b) has four. No transformation of knot shown in (a) may be made to turn it into the knot shown in (b) unless the knot is cut and rejoined. However (b) and (c) show representations of the same knot as the loop in (c) may be removed restoring the knot to its original figure eight form shown in (b).

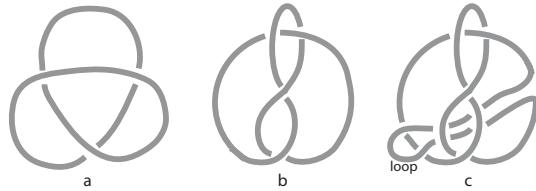


Figure 7 Trefoil knot (a), figure eight knot (b), Figure eight knot with extra loop (c).

Table 1. Summary of similarities and differences between textile knot practice and mathematical knot theory.

Property	Textile knot practice	Mathematical knot theory
Ends	May have loose ends	Continuous curve with no loose ends.
Material	Textile practice material dependent	Not concerned with materiality. Cross section of strand deemed to be a point.
Tension	Textile practice tension dependent, internal and external spaces important	A tight knot has the same representation as a loose knot so are considered equivalent.
Form	The addition of extra loops changes the appearance of a knot	If a knot may be simplified to the same representation as another knot, they are considered equivalent.

Table 1 shows that there are clear differences between textile knot practice and mathematical knot theory. On reflection however, in the consideration of Nimkulrat's textile practice, these differences are not significant and may be ignored for the following reasons: The same material is used consistently so there are no differences due to the choice of material. All strands have the same thickness. Knots are tied consistently with the same tension

and spacing throughout. Although more than one knot type is used (only one considered here), knots are tied in their simplest form. Additional loops or crossings are not introduced. The one area of difference that has been identified to date is the role of loose ends. In Nimkulrat's practice, knots are sometimes tied in combination so there are no loose ends. This aspect will now be considered further.

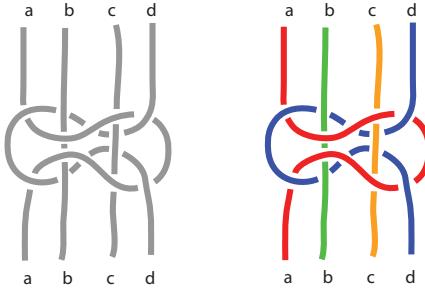


Figure 8 Left, diagram of knot (black) and right, diagram of knot (colored).

Figure 8 shows a diagrammatic representation of the knot described above (Figure 4) with loose ends. It is interesting to note that in coloring the diagram, one color per strand, that it becomes self-evident that the strands finish in the same positions that they started. This highlights that in this type of cross-disciplinary work that it is useful to develop a taxonomy, which may or may not be visual, to describe the textile practice, here the steps that are required to tie a knot in a certain way. The use of color reveals properties e.g., strand position, that are not otherwise obvious.

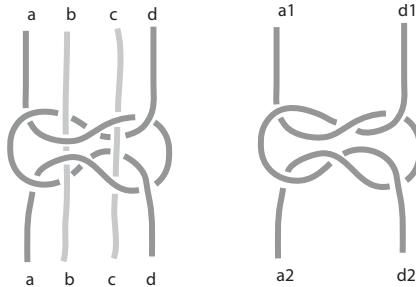


Figure 9 Left, diagram of knot (black) and right, diagram of knot with central strands b and c removed.

Given that a mathematical knot has by definition no loose ends, the role of loose ends on the knot in Figure 4 will now be explored. To simplify the analysis, strands b and c, which pass through the center of the knot are removed as they do not play a direct role in the tying of the knot (See Figure 9). They will be reintroduced later as they may play a role in the tying of sets of more than one knot e.g. to make a knot flower.

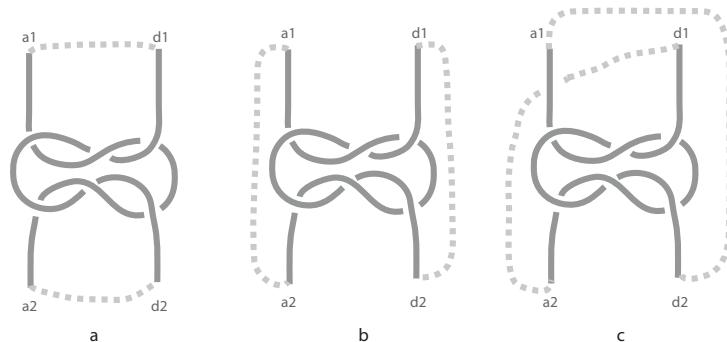


Figure 10 Dotted lines indicate where loose ends are joined. Three options (a, b, and c) are available.

To indicate the start and ends of strands a and d, the loose ends are relabeled a1 and a2 and d1 and d2 respectively. These may be joined in three different possible combinations namely:

- a1 joined to d1 and a2 joined to d2 (Figure 10a)
- a1 joined to a2 and d1 joined to d2 (Figure 10b)
- a1 joined to d2 and a2 joined to d1 (Figure 10c)

Two methods are used to investigate these knots further. The first is to add orientation arrows (Figure 11 middle). Add a start point on the curve of the knot (black circle). Trace the knot moving in a clockwise direction adding orientation arrows on each segment in the direction of travel until the start position is arrived at. The second method is to color the continuous curve (Figure 11 bottom). Using both methods, the diagrams show that mathematical knots are obtained in two cases. Figures 11a and 11c represent continuous curves (only one color) and comply with the definition of a mathematical knot. Figure 11b introduces a new concept – a *link* which is defined as a collection (set) of separate knots which are tangled together in such a way that they cannot be separated. A knot may be considered to be a link with only one component [1].

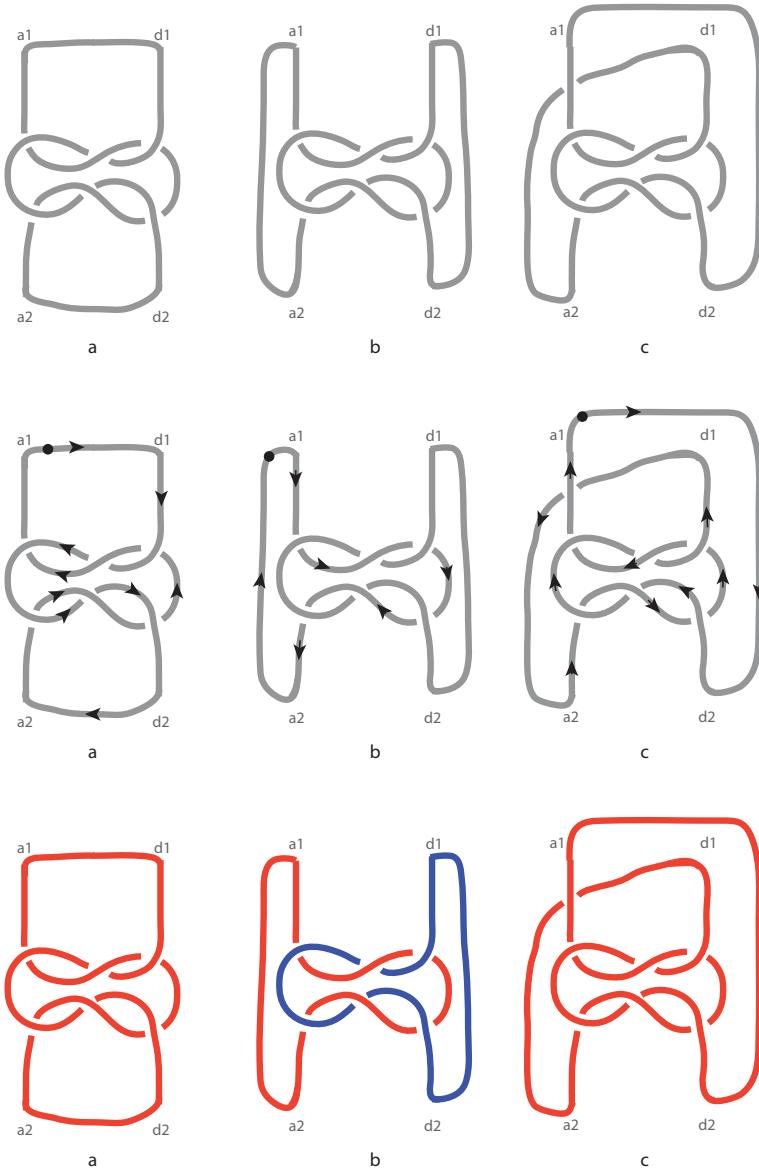


Figure 11: Use of orientation arrows and color to determine whether a curve is a knot or a series of links.

Returning to the knot described in Figure 8, the two withdrawn strands are replaced so that the knot now contains all four strands. The loose ends of the same strands are joined (start of a to end of a, start of b to end of b, etc.), it can be seen that the knot under discussion analyzed in this way represents not a knot with many crossings but a link with four components tangled together. Each component a, b, c, and d is a ring (Figure 12). Examining each ring individually, it may be seen that they do not contain crossings. Rings such as these are the simplest from of knot and are known as the trivial knot or the *unknot*.

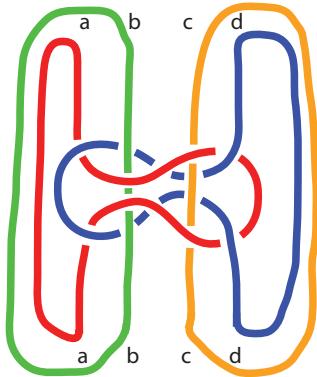


Figure 12 A link containing four components a, b, c, and d. Each component is a trivial knot.

Other alternatives such as joining different strands have not yet been considered and work is ongoing to analyze this textile practice in relation to mathematical braids and composite knots. Other knots used in Nimkulrat's practice will be analyzed and it is intended to develop the taxonomy and appropriate characterization methods further as research progresses.

5. Conclusions

This paper demonstrates that it is possible to explore one type of craft knot used in Nimkulrat's textile practice through the application of mathematical knot theory. The characterization reveals significant differences between craft knots and knots in mathematical knot theory. The examination of one knot property, loose ends led to significant insights. The research into other properties and knot types is ongoing. This characterization is one example of how knot theory may be used to examine the properties of knotted textile structures. The understanding of mathematical properties of craft knots may facilitate the practitioner's communication of creative process in a more objective and detailed way. This research illuminates how the problem of non-propositional contents of experiential knowledge embodied in a creative practitioner can be resolved with the support of mathematics that is adopted to examine a completed artifact. In this case, knot theory was used as a method to investigate, transfer and replicate the experiential and procedural knowledge of the knotting process, so that the content becomes explicit.

Mathematical models which reveal the nature of a particular knot types stimulate new ideas, which may not have occurred otherwise. They may be used to design a variety of knotted textile structures that are visually different from one another, yet adopt only a single type of knot in a topological sense. They may also help to plan the craft knotting process, for example the estimation of the length of strands required for textile production more precisely and the steps that are needed to tie a knot in a more accurate way. The value and impact of this

knowledge will be tested in Nimkulrat's subsequent knot textile practice. As textile practitioners, the researchers are aware of other properties of textile knots – material, tension, and form – that are as equally important as ends. Tension and material in particular are directly linked to the aesthetics of knotted textiles in terms of friction or the tightness or looseness of knots, and will be the property to be considered in the next step of examination. Theories in knot physics include models to calculate friction [14] may be considered in relation to material properties, as they may help to make knot theory more physically relevant to knotted physical objects [18]. A further aim of the next stage of this research will include widening the study into other textile and/or craft practices that involve repetitive structures. Research will also investigate the articulation of textile techniques and processes, e.g. knitting or patchwork algorithms.

The study to date shows the way in which the making of the knots in textile practice can be communicated more explicitly using methods from mathematical knot theory. It highlights the aesthetics of mathematics and the science of textile creation. Their relationship to the learning of mathematics will be explored in the latter stage of the research project.

6. References

- [1] Adams, C. (1994) *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots*, W.H. Freeman and Company, New York.
- [2] Åström, A. and Åström, C. (2011) *Circular knotworks consisting of pattern no. 295: a mathematical approach*, Journal of Mathematics and the Arts, vol. 5, no. 4, pp 185-197.
- [3] Bosch, R. (2010) *Simple-closed-curve sculptures of knots and links*, Journal of Mathematics and the Arts, vol. 4, no. 2, pp 57-71.
- [4] Devlin, K. (1999) *Mathematics: The New Golden*, 2nd ED., Columbia University Press, New York.
- [5] Friedman, N (2007) *Foreword*, Journal of Mathematics and the Arts, vol. 1, no. 1, pp 3-4.
- [6] Greenfield, G. (2007) *Editor's Introduction*, Journal of Mathematics and the Arts, vol. 1, no. 1, pp 1.
- [7] Harris, M. (1988) *Common Threads: Mathematics and Textiles*, Mathematics in School, vol. 17, no. 4, pp 24-28.
- [8] Harris, M. (1997) *Common Threads: Women, Mathematics and Work*, Trentham Books, Stoke-on-Trent.
- [9] Hart, G. W. (2002) *Solid-Segment Sculptures*, In C. P. Bruter (Ed.) *Mathematics and Art: Mathematical Visualization in Art and Education*, pp 17-27, Springer-Verlag, New York.
- [10] Issey Miyake Inc. (n.d.) *132 5. Issey Miyake* [online], accessed 19 March 2013 from: http://www.isseymiyake.com/en/brands/132_5.html
- [11] Jablanand, S. and Sazdanovic, R. (2007) *LinKnot: Knot theory by computer*, vol. 21, World Scientific, Singapore.
- [12] Küchler, S. (2001) Why knot? Towards a Theory of Art and Mathematics, In C. Pinney & N. Thomas (Eds.) *Beyond Aesthetics: Art and the Technologies of Enchantment*, pp 57-71, Berg, Oxford.
- [13] Lee, M. E. M. and Ockendon, H. (2005) *A Continuum Model for Entangled Fibres*, European Journal of Applied Mathematics, vol. 16, pp 145-160.

- [14] Meluzzi, D., Smith, D. E. and Arya, G. (2010) *Biophysics of Knotting*, Annual Review of Biophysics, no. 39, pp. 349-366.
- [15] Nimkulrat, N. (2009) *Paperness: Expressive Material in Textile Art from an Artist's Viewpoint*, University of Art and Design, Helsinki.
- [16] Ross, F. and Ross, W. T. (2011) *The Jordan curve theorem is non-trivial*, Journal of Mathematics and the Arts, vol. 5, no. 4, pp 213-219.
- [17] Sarhangi, R. (2007) *Foreword*, Journal of Mathematics and the Arts, vol 1, no. 1, pp 5-6.
- [18] Stasiak, A., Katritch V., and Kauffman, L. H. (Eds.) (1998) *Ideal Knots*, World Scientific, Singapore.
- [19] Sossinsky, A. (2002) *Knots: Mathematics with a Twist*, Harvard University Press, London.
- [20] Taimina, D. (2009) *Crocheting Adventures with Hyperbolic Planes*, AK Peters, Wellesley, MA.
- [21] Woodhouse, T. and Brand, A. (1920) *Textile Mathematics: Part I*, Blackie & Son, London.
- [22] Woodhouse, T. and Brand, A. (1921) *Textile Mathematics: Part 2*, Blackie & Son, London.